EFFICIENT SCHEMES ON SOLVING FRACTIONAL
INTEGRO-DIFFERENTIAL EQUATIONS

LOH JIAN RONG

A thesis submitted in
fulfillment of the requirement for the award of the
Doctor of Philosophy in Science

Faculty of Applied Sciences and Technology
Universiti Tun Hussein Onn Malaysia

FEBRUARY 2018

To Dr. Phang, UTHM and my family

ACKNOWLEDGEMENT

I am so grateful and greatly indebted to my supervisor Dr. Phang Chang for giving me the opportunity to pursue my research and providing me with great guidances and advices which gives me ideas, inspiration, motivation and confidence to continue the work. His patience, inspiration and guidance always lead me to positive outcomes and guided me in a prospective direction whenever I found myself caught in a bottleneck situation. Without his guidance, I would not be able to face the challenges, overcome various obstacles and hardship during these years and eventually accomplish the results obtained. I would like to thank Universiti Tun Hussein Onn Malaysia for giving me the one-lifetime chance for pursuing my dream to study mathematics, my beloved subject. I am also very grateful to the Ministry of Higher Education for granting me the MyBrain15 (MyPhd) financial support as well as Universiti Tun Hussein Onn Malaysia ORICC FRGS Grant during the initial stage of my study program which greatly aids any financial needs in research and daily expenses required. This greatly helps me to be able to focus fully on my research without having to worry over financial difficulties. Special thanks to my friend and classmate, Dr. Abdulnasir Isah, who inspire me with many ideas through many fruitful discussions. Discussions and ideas by him continue to motivate and inspire me with new possibilities. I would like to take this opportunity as well to thank my parents and my family members for their unconditional and continuing moral support which guided me to face various problems and dilemma in both personal life and research.

Loh Jian Rong

ABSTRACT

Fractional integro-differential equation (FIDE) emerges in various modelling of physical phenomena. In most cases, finding the exact analytical solution for FIDE is difficult or not possible. Hence, the methods producing highly accurate numerical solution in efficient ways are often sought after. This research has designed some methods to find the approximate solution of FIDE. The analytical expression of Genocchi polynomial operational matrix for left-sided and right-sided Caputo's derivative and kernel matrix has been derived. Linear independence of Genocchi polynomials has been proved by deriving the expression for Genocchi polynomial Gram determinant. Genocchi polynomial method with collocation has been introduced and applied in solving both linear and system of linear FIDE. The numerical results of solving linear FIDE by Genocchi polynomial are compared with certain existing methods. The analytical expression of Bernoulli polynomial operational matrix of right-sided Caputo's fractional derivative and the Bernoulli expansion coefficient for a two-variable function is derived. Linear FIDE with mixed left and right-sided Caputo's derivative is first considered and solved by applying the Bernoulli polynomial with spectral-tau method. Numerical results obtained show that the method proposed achieves very high accuracy. The upper bounds for the L^2 norm error of the N-order approximate solution for both Genocchi polynomial and Bernoulli polynomial are derived. In addition, Jacobi wavelet is defined and its operational matrix of Riemann-Liouville fractional integral is derived. Jacobi wavelet with spectral-tau method is proposed and applied to solve linear FIDE. Numerical results are obtained in terms of absolute error. Finally, a special linear time FPIDE has been introduced and its general analytical solution has been obtained using Laplace transform and classical resolvent kernel method. Case I and Case II approximate solutions by truncated Neumann series and truncated Taylor series are introduced. Upper bounds for the L^2 norm error estimates of Case I and Case II approximate solutions are derived. Illustrative examples of solving this special class of FPIDE using the proposed method are shown.

ABSTRAK

Persamaan pembezaan kamiran pecahan (PPKP) muncul dalam pelbagai model matematik untuk fenomena fizikal. Biasanya, penyelesaian analitik untuk persamaan ini susah diperolehi. Oleh itu, penyelesaian secara berangka menjadi penting. Kajian ini telah membina beberapa jenis kaedah berangka untuk memperolehi penyelesaian PPKP. Ungkapan analitik untuk matrik-operasi-pembezaan-pecahan-jenis-Caputo sebelah kiri dan sebelah kanan dan matrik kernel telah diperolehi. Ketakbersandaran linear untuk polinomial Genocchi telah dibuktikan dengan memperolehi ungkapan penentu Gram. Kaedah berdasarkan polinomial Genocchi bersama dengan cara kolokasi telah menyelesaikan secara berangka untuk PPKP linear dan sistem. Hasil kaedah ini dalam penyelesaian PPKP linear telah dibandingkan dengan kaedah lain. Ungkapan analitik untuk matrik-operasi-pembezaan-pecahan-sebelah-kanan jenis Caputo dan pekali pengembangan untuk fungsi dua pembolehubah berdasarkan polinomial Bernoulli telah diperolehi. PPKP-linear-bercampur pembezaan sebelah kiri dan kanan telah diperkenalkan dan diselesaikan dengan kaedah polinomial Bernoulli bersama kaedah spektrum-tau. Hasil penyelesaian berangka dengan kaedah ini telah diperolehi dengan tahap ketepatan yang tinggi. Batas atas untuk ralat norm L^2 telah diperolehi untuk polinomial Genocchi dan polinomial Bernoulli. Seterusnya, wavelet Jacobi telah ditakrifkan dan matrik-operasi-pengamiran-pecahan-jenis-Riemann-Liouville telah diperolehi. Kaedah wavelet Jacobi bergabung dengan kaedah spektrum-tau telah diperkenalkan untuk penyelesaian PPKP linear secara berangka. Hasil berangka telah diperolehi dalam bentuk ralat mutlak. Akhir sekali, sejenis persamaan pembezaan-separa-pecahan kamiran dalam pembolehubah masa diperkenalkan dan diselesaikan dengan menggunakan gabungan kaedah jelmaan Laplace dengan kaedah inti resolvent. Penyelesaian hampiran Kes I dan Kes II secara siri Neumann terpenggal dan siri Taylor terpenggal telah diperkenalkan. Batas atas untuk anggaran ralat norm L^2 untuk penyelesaian hampiran Kes I dan Kes II telah diperolehi. Beberapa contoh untuk jenis persamaan ini telah ditunjukkan bersama penyelesaian dengan kaedah yang diperkenalkan.

CONTENTS

	TITLE	**i**
	DECLARATION	**ii**
	DEDICATION	**iii**
	ACKNOWLEDGEMENT	**iv**
	ABSTRACT	**v**
	ABSTRAK	**vi**
	CONTENTS	**vii**
	LIST OF TABLES	**xvi**
	LIST OF FIGURES	**xxi**
	LIST OF SYMBOLS AND ABBREVIATIONS	**xxii**
	LIST OF APPENDICES	**xxix**
CHAPTER 1	**INTRODUCTION**	**1**
	1.1 Background	1
	1.1.1 Fractional Calculus	1
	1.1.2 Fractional Integro-Differential Equations and Fractional Partial Integro-Differential Equations	4
	1.2 Preliminaries and Basic Concepts	6
	1.2.1 Fractional Derivative and Integral:Riemann-Liouville and Caputo	6
	1.2.1.1 Riemann-Liouville	6

1.2.1.2 Caputo's Fractional Derivative with Singular Kernel 7

1.2.1.3 Basic Properties of Riemann-Liouville and Caputo 9

1.2.2 Laplace Transform for Fractional Calculus 11

1.2.3 Operational Matrix 14

1.2.4 Classical Resolvent Kernel Method for linear Volterra integral equation of second kind 15

1.3 Problem Statements 17

1.4 Objectives 19

1.5 Scope of Study 20

1.6 Main Contributions 22

1.7 Thesis Outline 23

CHAPTER 2 LITERATURE REVIEW 25

2.1 Fractional Integro-Differential Equations (FIDE) 26

2.1.1 Differential Transform and Integral Transform-based Methods 26

2.1.2 Variational Iteration Method (VIM) 27

2.1.3 Homotopy Perturbation based Methods(HPM) 28

2.1.4 Homotopy Analysis Method (HAM) 28

2.1.5 Series Methods 29

2.1.6 Polynomial-based Methods 29

2.1.7 Wavelet-based Methods 33

2.1.8 Other Methods 34

	2.2	Fractional Partial Integro-Differential Equation (FPIDE)	35
		2.2.1 Variational Iteration Method (VIM)	36
		2.2.2 Transform-based method	36
		2.2.3 Polynomial-based method	37
		2.2.4 Wavelet-based method	37
		2.2.5 Other methods	37
	2.3	Concluding Remarks	38
CHAPTER 3	**RESEARCH METHODOLOGY**		**39**
	3.1	Procedures of numerical solution of FIDE by polynomial-based method with collocation/spectral-tau	39
	3.2	Wavelet operational matrix of fractional integral with spectral-tau	44
	3.3	Procedures of solving a special time-linear Fractional Partial Integro-Differential Equation	45
	3.4	Computation of Error	46
	3.5	Concluding Remarks	47
CHAPTER 4	**COLLOCATION METHOD FOR SOLVING FIDE USING OPERATIONAL MATRIX VIA GENOCCHI POLYNOMIALS**		**48**
	4.1	Introduction	48
	4.2	Genocchi Polynomials: Basic Concepts	49
		4.2.1 Genocchi Polynomials: Definitions and Basic Properties	49
	4.3	Function approximation by Genocchi polynomial	52
		4.3.1 Computation of expansion coefficients for single-variable	

function approximation by Genocchi polynomials 52

4.3.2 On the Linear Independence of Genocchi Polynomials and Uniqueness of Approximation 56

4.3.3 Computation of Genocchi expansion coefficients for two-variable function 60

4.4 Genocchi Polynomial Operational Matrix of Caputo's Fractional Derivative 63

4.4.1 Derivation of Operational Matrix of Fractional Derivative 64

4.4.2 Genocchi Polynomial Operational Matrix of Left-Sided Caputo's Fractional Derivative 66

4.4.2.1 Left-Sided Caputo's Derivative Case (I): Operational Matrix of Fractional-Order $\alpha \neq n = \lceil \alpha \rceil$ 66

4.4.2.2 Left-Sided Caputo's Derivative Case (II): Operational Matrix of Integer-Order $\alpha = n = \lceil \alpha \rceil$ 69

4.4.3 Genocchi Polynomial Operational Matrix of Right-Sided Caputo's Fractional Derivative 71

4.4.3.1 Right-Sided Caputo's Derivative Case (I): Operational Matrix of Fractional-Order $\alpha \neq n = \lceil \alpha \rceil$ 71

4.4.3.2 Right-Sided Caputo's Derivative Case (II): Operational Matrix of Integer-Order $\alpha = n = \lceil \alpha \rceil$ 73

4.5 Error Analysis 75

4.5.1 Case 1:Solution known 75

4.5.2 Case 2: Solution unknown 76

4.5.3 Comparison of Relative Error of Genocchi polynomial operational matrix with other polynomial operational matrices 77

4.6 Numerical Solution of Fractional Integro-Differential Equation with Caputo's derivative by Genocchi polynomial approximation 79

4.6.1 General Procedure of Genocchi polynomial-based method with collocation for Single Variable FIDE with Caputo's fractional derivatives 80

4.6.2 General Procedure of Genocchi polynomial-based method with collocation for Single Variable System of FIDEs with Caputo's fractional derivative 83

4.7 Illustrative Examples for Single linear Fredholm-Volterra FIDE equation (4.65) 86

4.7.1 Single FIDE with Integer-Order Derivatives 86

4.7.2 Single FIDE with Fractional-Order Derivatives 97

4.7.2.1 Case: known exact solution 97

4.7.2.2 Case: unknown exact solution 101

4.8 Illustrative Examples for System of Linear Volterra FIDE equation (4.66) 106

4.8.1 System of FIDE with Integer-Order Derivatives 106

4.9 Concluding Remarks 120

CHAPTER 5 SPECTRAL TAU METHOD FOR SOLVING FIDE VIA BERNOULLI POLYNOMIAL 121

5.1 Introduction 121

5.2 Bernoulli Polynomials: Basic Concepts 122

5.2.1 Bernoulli Polynomials: Definitions and Basic Properties 122

5.3 Function approximation by Bernoulli polynomial 125

5.3.1 Computation of Bernoulli expansion coefficients for single-variable function 125

5.3.2 Computation of Bernoulli expansion coefficients for two-variable function 128

5.4 Bernoulli Polynomial Operational Matrix of Caputo's Fractional Derivative 133

5.4.1 Existing Approach for Derivation of Operational Matrix of Fractional Derivative 135

5.4.2 Bernoulli Polynomial Operational Matrix of Left-Sided Caputo's Fractional Derivative 136

5.4.2.1 Left-Sided Caputo's Derivative Case (I): Operational Matrix of

Fractional-Order $\alpha \neq n = \lceil \alpha \rceil$ 136

5.4.2.2 Left-Sided Caputo's Derivative Case (II): Operational Matrix of Integer-Order $\alpha = n = \lceil \alpha \rceil$ 139

5.4.3 Bernoulli Polynomial Operational Matrix of Right-Sided Caputo's Fractional Derivative 141

5.4.3.1 Right-Sided Caputo's Derivative Case (I): Operational Matrix of Fractional-Order $\alpha \neq n = \lceil \alpha \rceil$ 141

5.4.3.2 Right-Sided Caputo's Derivative Case (II): Operational Matrix of Integer-Order $\alpha = n = \lceil \alpha \rceil$ 144

5.5 Error Analysis 145

5.5.1 Relative Error of Bernoulli polynomial operational matrix of right-sided Caputo's derivative 145

5.5.2 Error Bound Estimates 147

5.5.3 Case 1: Solution known 147

5.5.4 Case 2: Solution unknown 149

5.6 Solving Fractional Integro-Differential Equation with Right-Sided Caputo's derivative by Bernoulli polynomial approximation 150

5.7 Numerical Examples 152

5.7.1 Case with Integer-Order Derivatives 152

	5.7.2	Case with Fractional-Order Derivatives	155
	5.7.2.1	Case: known exact solution	155
	5.7.2.2	Case: unknown exact solution	159
	5.8	Concluding Remarks	160
CHAPTER 6		**JACOBI WAVELET METHOD FOR FIDE**	**161**
	6.1	Introduction	161
	6.2	Jacobi Polynomials: Some Basics	162
	6.3	Function Approximation by Jacobi Wavelet	164
	6.3.1	Construction of Jacobi Wavelet from Generalized Shifted Jacobi Polynomial	164
	6.3.2	Definition of Jacobi Wavelet and Function Approximation	167
	6.4	Jacobi Wavelet Operational Matrix of Riemann-Liouville Fractional Integration	169
	6.4.1	Riemann-Liouville Fractional Integral of single Jacobi wavelet basis function	171
	6.5	Jacobi Wavelet Method for Numerical Solution of Fractional Integro-Differential Equation	176
	6.5.1	General Procedure of Jacobi Wavelet with spectral-tau method for Multi-Order Linear Fredholm FIDE	177
	6.5.2	Illustrative Examples	182
	6.5.2.1	Case I: equations of integer-order	182
	6.5.2.2	Case II: equation of fractional orders	190
	6.6	Concluding Remarks	202

CHAPTER 7 **LAPLACE TRANSFORM WITH RESOLVENT METHOD FOR TIME-FRACTIONAL PARTIAL INTEGRO-DIFFERENTIAL EQUATION OF CAPUTO-VOLTERRA** **203**

7.1 Linear Time-Fractional Partial Integro-Differential Equation of Caputo-Volterra 204

7.2 Linear Time-Fractional Partial Integro-Differential Equation of Caputo-Volterra and analytical solution by Laplace transform and Resolvent Kernel Method 205

7.3 Approximate Solution by truncated Neumann series and truncated Taylor series 209

7.4 Error Analysis for Approximation by truncated Neumann series and truncated Taylor series 210

7.4.1 Error Estimates 211

7.5 Illustrative Examples 217

7.6 Concluding Remarks 224

CHAPTER 8 **CONCLUSIONS AND FUTURE WORKS** **225**

8.1 Conclusions 225

8.2 Future Directions 229

REFERENCES **232**

APPENDICES A - D **243**

LIST OF PUBLICATIONS **255**

VITA

LIST OF TABLES

4.1 Comparison of relative error $e_{N;r}(x)$ between SLOMFD and GPOMFD with $\alpha = 0.5$ at points $x = 0.1, 0.6, 1$ 79

4.2 Example 4.7.2: Comparison of absolute errors between $N = 11$ Genocchi polynomial and $N = 10$ Bernoulli method (Bhrawy *et al.* (2012)) 91

4.3 Example 4.7.2: Comparison of absolute errors between $N = 11$ Genocchi polynomial with Improved HPM method (Yusufolu (Agadjanov)), CAS method (Danfu and Xufeng (2007)) and Differential Transform (Darania and Ebadian (2007)) 92

4.4 Example 4.7.2: Comparison of absolute errors between $N = 17$ Genocchi polynomial and $N = 16$ Bernoulli method (Bhrawy *et al.* (2012)) 95

4.5 Example 4.7.3: Comparison of L^{∞} maximum absolute error between orders $N = 7, 8, 9$ Genocchi polynomials and that of Bernoulli method (Bhrawy *et al.* (2012)) 97

4.6 Example 4.7.4: Comparison of approximate solutions and absolute errors between order $N = 8$ Genocchi polynomials and $(p, M) = (4, 2)$ Legendre wavelet method (LWM) ((Meng et al., 2015)) 98

4.7 Example 4.7.4: Comparison of absolute errors

	between order $N = 8$ Genocchi polynomials and $N = 8$ Euler, $N = 8$ Fibonacci, $N = 8$ Bernoulli, $N = 7$ Bernstein	99
4.8	Example 4.7.5: Comparison of absolute errors between order $N = 8$ Genocchi polynomials and $(p, M) = (3, 2)$ Legendre wavelet method (LWM) of (Meng et al. (2015))	100
4.9	Example 4.7.5: Comparison of absolute errors between order $N = 8$ Genocchi polynomials and $N = 8$ Euler, $N = 8$ Fibonacci, $N = 8$ Bernoulli, $N = 7$ Bernstein	101
4.10	Example 4.7.6 : L^2-norm error ϵ_N for $N = 4, 5, 6, 7, 8$ Genocchi polynomials	102
4.11	Example 4.7.6 : N-level error estimate $\Delta_N(x)$ for $N = 4, 5, 6, 7, 8$ Genocchi polynomials	103
4.12	Example 4.7.7 : L^2-norm error ϵ_N for $N = 4, 5, 6, 7, 8$ Genocchi polynomials	104
4.13	Example 4.7.7 : N-level error estimate $\Delta_N(x)$ for $N = 4, 5, 6, 7, 8$ Genocchi polynomials	104
4.14	Example 4.7.8 : L^2-norm error ϵ_N for $N = 4, 5, 6, 7, 8$ Genocchi polynomials	105
4.15	Example 4.7.8 : N-level error estimate $\Delta_N(x)$ for $N = 4, 5, 6, 7, 8$ Genocchi polynomials	106
4.16	Example 4.8.2 Mirzaee and Bimesl (2014): Comparison of absolute errors $e_{1,N}$ between $N = 8$ Genocchi polynomial approximate solution $y^*_{1,8}(x)$ and that for $N = 7$ Euler polynomial $y^*_{1,7}(x)$ by Mirzaee and Bimesl (2014)	111
4.17	Example 4.8.2 Mirzaee and Bimesl (2014): Comparison of absolute errors $e_{2,N}$ between $N =$ 8Genocchi polynomial approximate solution $y^*_{2,8}(x)$ and that for $N = 7$ Euler polynomial $y^*_{2,7}(x)$ by Mirzaee and Bimesl (2014)	111

4.18 Example 4.8.2 (Mirzaee and Bimesl (2014)): Comparison of maximum absolute error $e_{1,N}^{\infty}$ between $N = 4, 6, 8, 11$ Genocchi polynomial approximate solutions $y_{1,N}^{*}(x)$ and that for $N = 3, 5, 7, 8, 10$ Euler method approximate solutions 112
4.19 Example 4.8.2 (Mirzaee and Bimesl (2014)): Comparison of maximum absolute error $e_{2,N}^{\infty}$ between $N = 4, 6, 8, 11$ Genocchi polynomial approximate solutions $y_{2,N}^{*}(x)$ and that for $N = 3, 5, 7, 8, 10$ Euler method approximate solutions 112
4.20 Example 4.8.3 (Sekar and Murugesan (2016)): Relative error $e_{N;r}(x)$ of approximate $y_{1,8}^{*}(x)$ of $N = 8$ Genocchi polynomials method with respect to $m = 200$ by Single Term Walsh Series (STWS) 114
4.21 Example 4.8.3 (Sekar and Murugesan (2016)): Relative error $e_{N;r}(x)$ of approximate $y_{2,8}^{*}(x)$ of $N = 8$ Genocchi polynomials method with respect to $m = 200$ by Single Term Walsh Series (STWS) 114
4.22 Example 4.8.4 : Comparison of relative error $e_{N;r}(x)$ of approximate $y_{1,8}^{*}(x)$ of $N = 8$ Genocchi polynomials method with respect to $h = 4, j = 33$ biorthogonal system by (Berenguer et al. (2013)) 116
4.23 Example 4.8.4 : Comparison of relative error $e_{N;r}(x)$ of approximate $y_{2,8}^{*}(x)$ of $N = 8$ Genocchi polynomials method with respect to $h = 4, j = 33$ biorthogonal system by (Berenguer et al. (2013)) 116
4.24 Example 4.8.4 : Comparison of relative error $e_{N;r}(x)$ of approximate $y_{1,8}^{*}(x)$ of $N = 8$ Genocchi polynomials method with respect to

$h = 4, j = 33$ biorthogonal system by (Berenguer et al. (2013)) 117

4.25 Example 4.8.4 : Comparison of relative error $e_{N;r}(x)$ of approximate $y^*_{2,8}(x)$ of $N = 8$ Genocchi polynomials method with respect to $h = 4, j = 33$ biorthogonal system by (Berenguer et al. (2013)) 117

4.26 Example 4.8.5 (Mirzaee and Hoseini (2014)): Comparison of maximum absolute error $e^{\infty}_{1,N}$ between $N = 7, 8, 9, 10, 11, 12$ Genocchi polynomial approximate solutions $y^*_1(x)$and that of $N = 7, 8, 9, 10, 11, 12$ Fibonacci polynomial approximate solutions 119

4.27 Example 4.8.5 (Mirzaee and Hoseini (2014)): Comparison of maximum absolute error $e^{\infty}_{2,N}$ between $N = 7, 8, 9, 10, 11, 12$ Genocchi polynomial approximate solutions $y^*_2(x)$and that of $N = 7, 8, 9, 10, 11, 12$ Fibonacci polynomial approximate solutions 120

5.1 Example 5.7.5: L^2-norm error ϵ_N for $N = 4, 5, 6$ of Bernoulli polynomials 160

6.1 Example 6.5.4: Exact $f(x)$, Approximate $f^*(x)$ with $(p, M) = (1, 3)$,$(a, b) = (-1/2, -1/2)$ and absolute error 191

6.2 Example 6.5.5: Exact $f(x)$, Approximate $f^*(x)$ with $(p, M) = (1, 6)$,$(a, b) = (-1/2, -1/2)$ and absolute error 193

6.3 Example 6.5.6: Exact $f(x)$, Approximate $f^*(x)$ with $(p, M) = (1, 6)$,$(a, b) = (1/2, -1/2)$ and absolute error 195

6.4 Example 6.5.6: Comparison of absolute error of approximate solution approximate $f^*(x)$ between Jacobi wavelet $(p, M) = (1, 6)$,$(a, b) =$

	$(1/2,-1/2)$ and $N=6$ Genocchi collocation method, $N=6$ Bernoulli spectral-tau method	197
6.5	Example 6.5.6: Comparison of absolute error of approximate solution approximate $f^*(x)$ between Jacobi wavelet $(p,M)=(1,6)$,$(a,b)=(1/2,-1/2)$ and $N=6$ Euler, $N=6$ Fibonacci, $N=6$ Bernstein methods	198
6.6	Example 6.5.7: Comparison of absolute error of approximate solution approximate $f^*(x)$ between Jacobi wavelet $(a,b)=(1/2,1/2)$ and $N=4,8$ Genocchi collocation method, $N=4,8$ Bernoulli spectral-tau method	200
6.7	Example 6.5.8: Comparison of absolute error of approximate solution approximate $f^*(x)$ between Jacobi wavelet $(a,b)=(2/3,1/3)$ and $N=9$ Genocchi collocation method, $N=9$ Bernoulli spectral-tau method	202

LIST OF FIGURES

4.1	Comparison of relative error $e_{N;r}(x)$ of operational matrix of fractional derivative (OMFD) with $\alpha = 0.5$ between $N = 9$ Shifted Legendre polynomials and $N = 10$ Genocchi polynomials	78
4.2	Example 4.8.2 Mirzaee and Bimesl (2014): Graph of exact solution $y_1(x) = e^{2x}$ and Genocchi approximate solutions $y_{1,N}^*(x)$ for $N = 4, 8$	110
4.3	Example 4.8.2 Mirzaee and Bimesl (2014): Graph of exact solution $y_2(x) = e^{-2x}$ and Genocchi approximate solutions $y_{2,N}^*(x)$ for $N = 4, 8$	110
5.1	Comparison of relative error $e_{N;r}(x)$ of operational matrix of right-sided Caputo's fractional derivative for $\alpha = 0.5$ and $N = 5$ between Shifted Legendre polynomial (solid red line) and Bernoulli polynomial (dotted black line)	146

LIST OF SYMBOLS AND ABBREVIATIONS

$AC([a,b])$	real space of absolutely continuous functions over the finite interval $[a,b]$, $a < b$
$AC^n([a,b])$	real space of n-absolutely differentiable functions over the finite interval $[a,b]$, $a < b$
$AC^n(\mathbb{R})$	real space of n-absolutely differentiable functions over the real line $\mathbb{R}$
A	operator that acts on $h(x,t)$ by $\sum_{k=0}^{m-1} \frac{t^k}{\Gamma(k+1)} g_k(x) + I_{0+,t}^{\alpha}(h(x,t))$
A^T	Transpose of an arbitrary matrix A
$BesselI(v,x)$	Bessel function of the first kind of order $\boldsymbol{v}$
$B_n(x)$	Bernoulli polynomial of degree n
$\mathbf{B}(x)$	Bernoulli polynomial vector
b_n	Bernoulli number
$C([a,b])$	real space of continuous functions over the interval $[a,b]$
$C(\mathbb{R})$	real space of continuous functions over the real line $\mathbb{R}$
C	coefficient vector or expansion coefficient vector
$C^n(\mathbb{R})$	real space of n-differentiable functions over the real line $\mathbb{R}$
$C^\infty(I)$	real space of infinitely-differentiable (smooth) functions over the interval I □ $\mathbb{R}$
${}^{C}D_{a+}^{\alpha}$	Left-Sided Caputo fractional derivative over $[a,b]$
${}^{C}D_{b-}^{\alpha}$	Right-Sided Caputo fractional derivative over $[a,b]$
${}^{C}D_{0+,t}^{\alpha}$	Left-Sided Caputo partial fractional derivative over $[0,b]$

${}^{*}D_{*}^{\alpha}$	Left-Sided or Right-Sided fractional derivatives for any kind of definitions such as Riemann-Liouville, Caputo.
${}^{C}D_{*}^{\alpha}$	Left-Sided ${}^{C}D_{0+}^{\alpha}$ or Right-Sided Caputo's fractional derivatives ${}^{C}D_{1-}^{\alpha}$
D_{a+}^{α}	Left-Sided Riemann-Liouville fractional derivative over $[a,b]$
$dist_{L^2}(u,v;\Omega)$	distance function in L^2 norm between function $u(x,t)$ and $v(x,t)$ over the region Ω
d_i	The value of the initial condition of the i-th derivative of the unknown function, $f^{i}(0)$
$\mathbf{E}_{\alpha_u;\alpha_r}$	Jacobi wavelet coefficient vector of the summation series formed by initial values of Caputo's derivative of unknown function, ${}^{C}D_{0+}^{\alpha_r}f(0)$
$\mathbf{erf}(t)$	error function $\frac{2\int_0^x e^{-t^2}dt}{\sqrt{\pi}}$
e_N^{∞}	maximum absolute error function for N-order approximation of $f(x)$
$e_{1,N}^{\infty}$	maximum absolute error function for N-order approximation of $y_1(x)$
$e_{2,N}^{\infty}$	maximum absolute error function for N-order approximation of $y_2(x)$
$e_N(x)$	absolute error function for N-order approximation
$e_{N;L^2}$	error in L^2 norm of N-order approximation over the interval [0,1]
$e_{N;r}(x)$	relative error function for N-order approximation
FDE	Fractional Differential Equation
FIDE	Fractional Integro-Differential Equation
FPDE	Fractional Partial Differential Equation
FPIDE	Fractional Partial Integro-Differential Equation
$f(x)$	unknown exact solution of a linear FIDE
$\lvert f(x)\rvert$	absolute value of function $f(x)$
$\lVert f(x)\rVert$	norm of function $f(x)$

$\|f(x)\|_2$	L^2 norm of function $f(x)$
$f^{(i)}(0)$	initial value of i-th derivative of $f(x)$
$f_N^*(x)$	approximate solution using N order of the basis polynomials
$f_{N,BES}^*(x)$	approximate solution using N order of the Bernstein polynomials
$f_{N,BR}^*(x)$	approximate solution using N order of the Bernoulli polynomials
$f_{N,E}^*(x)$	approximate solution using N order of the Euler polynomials
$f_{N,FB}^*(x)$	approximate solution using N order of the Fibonacci polynomials
$f_{N,G}^*(x)$	approximate solution using N order of the Genocchi polynomials
$f_{p,M}^*(x)$	approximate solution using wavelets with parameter
$G_n(x)$	Genocchi polynomial of degree n
$\mathbf{G}(x)$	Genocchi polynomial vector
$\mathcal{G}_B(x,t)$	Generating Function of Bernoulli polynomials
$\mathcal{G}_b(t)$	Generating Function of Bernoulli numbers
$\mathcal{G}_G(x,t)$	Generating Function of Genocchi polynomials
$g_k(x)$	functions of boundary conditions of TFPIDE
g_n	Genocchi number
$H(x,s)$	Laplace transform of $h(\boldsymbol{x},t)$ with respect to t domain
$\mathbb{H}(t)$	Heaviside unit step function
$h(x)$	extra coefficient function for single-variable FIDE
$h(x,t)$	extra coefficient function for two-variable TFPIDE
$h_m^{(a,b,L)}$	orthogonality constant of Jacobi wavelet $\psi_{n,m}^{(a,b,L)}(x)$ of degree m, order (a, b)
$H_{M,apx}(x,s)$	approximate solution of $H(x,s)$ by M-order truncated Taylor series
$h_{N,M,apx}(x,t)$	inverse Laplace transform of $H_{N,M,apx}(x,s)$ with respect to s domain

IDE	Integro-Differential Equation
I^n	integer-order integral of order n
I_{a+}^{α}	Left-Sided Riemann-Liouville fractional integral over $[a,b]$
$K(x,t)$	integral kernel function of either Volterra type or Fredholm type
$\mathbf{K}^r$	r-iterated kernel operator
$K_F(x,t)$	integral kernel function of Fredholm type
$K_V(x,t)$	integral kernel function of Volterra type
$\mathbf{K}_{-;F}$	Fredholm integral kernel matrix
$\mathbf{K}_{-;v}$	Volterra integral kernel matrix
$L^2([a,b])$	L^2 normed space over the interval $[a,b]$
$L^\infty([a,b])$	L^∞ normed space over the interval $[a,b]$
$\mathcal{L}_s^{-1}$	Inverse Laplace transform with respect to s variable
$\mathcal{L}_t$	Laplace transform with respect to t variable
$mod(i,M)$	the modulo function which returns the remainder of $\frac{i}{M}$
$m!$	m factorial
(N,M)	order of approximation by N-order truncated Neumann series and M-order truncated Taylor series for TFPIDE of Caputo-Volterra type
$\mathbb{N}_0$	set of natural numbers $0, 1, 2, \ldots$
$(n)^{(k)}$	falling factorial, $n(n-1)\ldots(n-k+1)$
$(n)_{(k)}$	rising factorial, $n(n+1)\ldots(n+k-1)$
$\mathbf{P}_{*,B}^{\alpha}$	Bernoulli polynomial operational matrix of left or right sided Caputo's fractional derivative
$\mathbf{P}_{+,B}^{\alpha}$	Bernoulli polynomial operational matrix of left-sided Caputo's fractional derivative
$\mathbf{P}_{-,B}^{\alpha}$	Bernoulli polynomial operational matrix of right-sided Caputo's fractional derivative
$\mathbf{P}_{*,G}^{\alpha}$	Genocchi polynomial operational matrix of left or right sided Caputo's fractional derivative

REFERENCES

Abbasbandy, S., Hashemi, M. and Hashim, I. (2013). On convergence of homotopy analysis method and its application to fractional integro-differential equations. *Quaestiones Mathematicae*. 36(1), 93–105.

Adibi, H. and Assari, P. (2010). Chebyshev wavelet method for numerical solution of Fredholm integral equations of the first kind. *Mathematical problems in Engineering*. 2010.

Aminikhah, H. (2011). A new analytical method for solving systems of linear integro-differential equations. *Journal of King Saud University-Science*. 23(4), 349–353.

Angstmann, C., Henry, B. and McGann, A. (2016a). A fractional order recovery SIR model from a stochastic process. *Bulletin of mathematical biology*. 78(3), 468–499.

Angstmann, C. N., Henry, B. I. and McGann, A. V. (2016b). A fractional-order infectivity SIR model. *Physica A: Statistical Mechanics and its Applications*. 452, 86–93.

Araci, S. (2012). Novel Identities for q-Genocchi Numbers and Polynomials. *Journal of Function Spaces and Applications*. 2012.

Araci, S. (2014). Novel identities involving Genocchi numbers and polynomials arising from applications of umbral calculus. *Applied Mathematics and Computation*. 233, 599–607.

Araci, S., Acikgoz, M. and Şen, E. (2012). Some New Identities of Genocchi Numbers and Polynomials involving Bernoulli and Euler polynomials. *arXiv preprint arXiv:1209.0628*.

Askey, R. (1975). *Orthogonal polynomials and special functions*. SIAM.

Babolian, E. and Fattahzadeh, F. (2007). Numerical solution of differential equations by using Chebyshev wavelet operational matrix of integration. *Applied Mathematics and Computation*. 188(1), 417–426.

Bagley, R. L. and Torvik, J. (1983). Fractional calculus-a different approach to the analysis of viscoelastically damped structures. *AIAA journal*. 21(5), 741–748.

Bahşı, M. M., Bahşı, A. K., Çevik, M. and Sezer, M. (2016). Improved Jacobi matrix method for the numerical solution of Fredholm integro-differential-difference equations. *Mathematical Sciences*. 10(3), 83–93.

Balcı, M. A. and Sezer, M. (2016). Hybrid Euler–Taylor matrix method for solving of generalized linear Fredholm integro-differential difference equations. *Applied Mathematics and Computation*. 273, 33–41.

Barikbin, Z. (2017). Two-dimensional Bernoulli wavelets with satisfier function in the Ritz–Galerkin method for the time fractional diffusion-wave equation with damping. *Mathematical Sciences*, 1–8.

Beals, R. and Wong, R. (2010). *Special functions: a graduate text*. vol. 126. Cambridge University Press.

Behroozifar, M. and Sazmand, A. (2017). An approximate solution based on Jacobi polynomials for time-fractional convection–diffusion equation. *Applied Mathematics and Computation*. 296, 1–17.

Berenguer, M., Garralda-Guillem, A. and Galán, M. R. (2013). An approximation method for solving systems of Volterra integro-differential equations. *Applied Numerical Mathematics*. 67, 126–135.

Bhrawy, A. and Alofi, A. (2013). The operational matrix of fractional integration for shifted Chebyshev polynomials. *Applied Mathematics Letters*. 26(1), 25–31.

Bhrawy, A., Doha, E., Ezz-Eldien, S. and Abdelkawy, M. (2016a). A numerical technique based on the shifted Legendre polynomials for solving the time-fractional coupled KdV equations. *Calcolo*. 53(1), 1–17.

Bhrawy, A., Tharwat, M. and Alghamdi, M. (2014). A new operational matrix of fractional integration for shifted Jacobi polynomials. *Bulletin of the Malaysian Mathematical Sciences Society*. 37(4), 983–995.

Bhrawy, A., Tohidi, E. and Soleymani, F. (2012). A new Bernoulli matrix method for solving high-order linear and nonlinear Fredholm integro-differential equations with piecewise intervals. *Applied Mathematics and Computation*. 219(2), 482–497.

Bhrawy, A., Zaky, M. and Van Gorder, R. A. (2016b). A space-time Legendre spectral tau method for the two-sided space-time Caputo fractional diffusion-wave equation. *Numerical Algorithms*. 71(1), 151–180.

Biazar, J. and Ebrahimi, H. (2012). Chebyshev wavelets approach for nonlinear systems of Volterra integral equations. *Computers & Mathematics with Applications*. 63(3), 608–616.

Bushnaq, S., Maayah, B. and Ahmad, M. (2016). Reproducing Kernel Hilbert Space Method For Solving Fredholm Integro-Differential Equations Of Fractional Order. *Italian Journal of Pure and Applied Mathematics*. (36), 307–318.

Chang, P. and Isah, A. (2016). Legendre Wavelet Operational Matrix of fractional Derivative through wavelet-polynomial transformation and its Applications in Solving Fractional Order Brusselator system. In *Journal of Physics: Conference Series*, vol. 693. IOP Publishing, 012001.

Chatterjee, A., Basu, U. and Mandal, B. (2017). Numerical Solution of Volterra Type Fractional Order Integro-Differential Equations in Bernstein Polynomial Basis. *Applied Mathematical Sciences*. 11(6), 249–264.

Chen, C. and Hsiao, C. (1997). Haar wavelet method for solving lumped and distributed-parameter systems. *IEE Proceedings-Control Theory and Applications*. 144(1), 87–94.

Chen, Y., Ke, X. and Wei, Y. (2015). Numerical algorithm to solve system of nonlinear fractional differential equations based on wavelets method and the error analysis. *Applied Mathematics and Computation*. 251, 475–488.

Danfu, H. and Xufeng, S. (2007). Numerical solution of integro-differential equations by using CAS wavelet operational matrix of integration. *Applied Mathematics and Computation*. 194(2), 460–466.

Darania, P. and Ebadian, A. (2007). A method for the numerical solution of the integro-differential equations. *Applied Mathematics and Computation*. 188(1), 657 –

668.

Delves, L. M. and Mohamed, J. (1988). *Computational methods for integral equations*. CUP Archive.

Dhunde, R. R. and Waghmare, G. (2015). Solving Partial Integro-Differential Equations Using Double Laplace Transform Method. *American Journal of Computational and Applied Mathematics*. 5(1), 7–10.

Doha, E. (2004). On the construction of recurrence relations for the expansion and connection coefficients in series of Jacobi polynomials. *Journal of Physics A: Mathematical and General*. 37(3), 657.

Doha, E., Bhrawy, A. and Ezz-Eldien, S. (2011). A Chebyshev spectral method based on operational matrix for initial and boundary value problems of fractional order. *Computers & Mathematics with Applications*. 62(5), 2364–2373.

Doha, E., Bhrawy, A. and Ezz-Eldien, S. (2012). A new Jacobi operational matrix: An application for solving fractional differential equations. *Applied Mathematical Modelling*. 36(10), 4931–4943.

Doha, E. H., Bhrawy, A., Baleanu, D. and Ezz-Eldien, S. (2013). On shifted Jacobi spectral approximations for solving fractional differential equations. *Applied Mathematics and Computation*. 219(15), 8042–8056.

Ebrahimi, N. and Rashidinia, J. (2014). Spline collocation for Fredholm and Volterra integro-differential equations. *International Journal of Mathematical Modeling and Computations*. 4(3), 289–298.

Erjaee, G., Akrami, M. and Atabakzadeh, M. (2013). The operational matrix of fractional integration for shifted Legendre polynomials. *Iranian Journal of Science and Technology (Sciences)*. 37(4), 439–444.

Eslahchi, M., Dehghan, M. and Parvizi, M. (2014). Application of the collocation method for solving nonlinear fractional integro-differential equations. *Journal of Computational and Applied Mathematics*. 257, 105–128.

Ezzati, R., Gholam, A. M. and Nouriani, H. (2015). Numerical solution for nth order linear Fredholm integro-differential equations by using Chebyshev wavelets integration operational matrix.

Fitt, A., Goodwin, A., Ronaldson, K. and Wakeham, W. (2009). A fractional differential equation for a MEMS viscometer used in the oil industry. *Journal of Computational and Applied Mathematics*. 229(2), 373–381.

Gorenflo, R. and Mainardi, F. (2008). Fractional calculus: integral and differential equations of fractional order, 2008. *arXiv preprint arXiv:0805.3823*.

He, J.-H. (1999). Variational iteration method–a kind of non-linear analytical technique: some examples. *International journal of non-linear mechanics*. 34(4), 699–708.

He, J.-H. (2003). Homotopy perturbation method: a new nonlinear analytical technique. *Applied Mathematics and computation*. 135(1), 73–79.

He, J.-H. (2007). Variational iteration methodsome recent results and new interpretations. *Journal of computational and applied mathematics*. 207(1), 3–17.

Horadam, A. (1991). Genocchi polynomials. *Proceedings of the 4th International Conference on Fibonacci Numbers and Their Applications*, 145–166.

Huang, L., Li, X. F., Zhao, Y. and Duan, X. Y. (2011). Approximate solution of fractional integro-differential equations by Taylor expansion method. *Computers & Mathematics with Applications*. 62(3), 1127–1134.

Hussain, A. K., Fadhel, F. S., Yahya, Z. R. and Rusli, N. (2016a). Variational Iteration Method (VIM) For Solving Partial Integro-Differential Equations. *Journal of Theoretical and Applied Information Technology*. 88(2), 367.

Hussain, A. K., Rusli, N., Fadhel, F. S., Yahya, Z. R., Rusli, N., Zaimi, W. M. K. A. W., Khazali, K. A. M., Masnan, M. J., Daud, W. S. W., Abdullah, N. *et al.* (2016b). Solution of one-dimensional fractional order partial integro-differential equations using variational iteration method. In *AIP Conference Proceedings*, vol. 1775. AIP Publishing, 030096.

Inokuti, M., Sekine, H. and Mura, T. (2013). General use of the Lagrange multiplier in nonlinear mathematical physics. *Variational method in the mechanics of solids*. 33(5), 156–162.

Isah, A. and Phang, C. (2017). Operational matrix based on Genocchi polynomials for solution of delay differential equations. *Ain Shams Engineering Journal*.

Isah, A., Phang, C. and Phang, P. (2017). Collocation Method Based on Genocchi Operational Matrix for Solving Generalized Fractional Pantograph Equations. *International Journal of Differential Equations*. 2017.

Kajani, M. T., Vencheh, A. H. and Ghasemi, M. (2009). The Chebyshev wavelets operational matrix of integration and product operation matrix. *International Journal of Computer Mathematics*. 86(7), 1118–1125.

Kazem, S. (2013). Exact solution of some linear fractional differential equations by Laplace transform. *International Journal of Nonlinear Science*. 16(1), 3–11.

Keshavarz, E., Ordokhani, Y. and Razzaghi, M. (2016). A numerical solution for fractional optimal control problems via Bernoulli polynomials. *Journal of Vibration and Control*. 22(18), 3889–3903.

Kilbas, A., Marichev, O. and Samko, S. (1993). Fractional Integral and Derivatives (Theory and Applications). *Gordon and Breach, Switzerland*. 1(993), 1.

Kilbas, A., Srivastava, H. and Trujillo, J. (2006). *Theory and Applications of Fractional Differential Equations*. vol. 204. Amsterdam: Elsevier.

Kiryakova, V. S. (1993). *Generalized fractional calculus and applications*. CRC press.

Kreyszig, E. (1989). *Introductory functional analysis with applications*. vol. 1. wiley New York.

Kumar, K., Pandey, R. K. and Sharma, S. (2017). Comparative study of three numerical schemes for fractional integro-differential equations. *Journal of Computational and Applied Mathematics*. 315, 287–302.

Kurt, B. (2010). A further generalization of the Bernoulli polynomials and on the 2D-Bernoulli polynomials $B_n^2(x, y)$. *Appl. Math. Sci.* 4(47), 2315–2322.

Li, Y. and Zhao, W. (2010). Haar wavelet operational matrix of fractional order integration and its applications in solving the fractional order differential equations. *Applied Mathematics and Computation*. 216(8), 2276–2285.

Liao, S. (1997). Homotopy analysis method: a new analytical technique for nonlinear problems. *Communications in Nonlinear Science and Numerical Simulation*.

2(2), 95–100.

Liao, S. J. (1992). *The proposed homotopy analysis technique for the solution of nonlinear problems*. Ph.D. Thesis. Ph. D. Thesis, Shanghai Jiao Tong University.

Liu, J., Li, X. and Wu, L. (2016). An Operational Matrix Technique for Solving Variable Order Fractional Differential-Integral Equation Based on the Second Kind of Chebyshev Polynomials. *Advances in Mathematical Physics*. 2016.

Loverro, A. (2004). Fractional calculus: history, definitions and applications for the engineer. *Rapport technique, Univeristy of Notre Dame: Department of Aerospace and Mechanical Engineering*, 1–28.

Mahdy, A. M. and Shwayyea, R. T. (2016). Numerical solution of fractional integro-differential equations by least squares method and shifted Laguerre polynomials pseudo-spectral method. *International Journal of Scientific & Engineering Research*. 7(4).

Mahgob, M. M. A. (2015). Solution of partial integro-differential equations by double Elzaki transform method. *Mathematical Theory and Modeling*. 5(5).

Maleknejad, K., Basirat, B. and Hashemizadeh, E. (2012). A Bernstein operational matrix approach for solving a system of high order linear Volterra–Fredholm integro-differential equations. *Mathematical and Computer Modelling*. 55(3), 1363–1372.

Marcellán, F. and Van Assche, W. (2006). *Orthogonal polynomials and special functions: computation and applications*. 1883. Springer Science & Business Media.

Meng, Z., Wang, L., Li, H. and Zhang, W. (2015). Legendre wavelets method for solving fractional integro-differential equations. *International Journal of Computer Mathematics*. 92(6), 1275–1291.

Miller, K. S. and Ross, B. (1993). An introduction to the fractional calculus and fractional differential equations.

Mirzaee, F. and Bimesl, S. (2014). A new Euler matrix method for solving systems of linear Volterra integral equations with variable coefficients. *Journal of the*

Egyptian Mathematical Society. 22(2), 238–248.

Mirzaee, F. and Hoseini, S. F. (2014). Solving systems of linear Fredholm integro-differential equations with Fibonacci polynomials. *Ain Shams Engineering Journal*. 5(1), 271–283.

Mokhtary, P. and Ghoreishi, F. (2011). The L2-convergence of the Legendre spectral Tau matrix formulation for nonlinear fractional integro differential equations. *Numerical Algorithms*. 58(4), 475–496.

Mustafa Gülsu, Yalçn Öztürk and Ayşe Anapal (2013). Numerical approach for solving fractional Fredholm integro-differential equation. *International Journal of Computer Mathematics*. 90(7), 1413–1434.

Nawaz, Y. (2011). Variational iteration method and homotopy perturbation method for fourth-order fractional integro-differential equations. *Computers & Mathematics with Applications*. 61(8), 2330–2341.

Odibat, Z. and Momani, S. (2009). The variational iteration method: an efficient scheme for handling fractional partial differential equations in fluid mechanics. *Computers & Mathematics with Applications*. 58(11), 2199–2208.

Pedas, A., Tamme, E. and Vikerpuur, M. (2016). Spline collocation for fractional weakly singular integro-differential equations. *Applied Numerical Mathematics*. 110, 204–214.

Podlubny, I. (1998). *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. vol. 198. Academic press.

Rad, J., Kazem, S., Shaban, M. and Parand, K. (2014). A new operational matrix based on Bernoulli polynomials. *arXiv preprint arXiv:1408.2207*.

Rahimkhani, P., Ordokhani, Y. and Babolian, E. (2017). A new operational matrix based on Bernoulli wavelets for solving fractional delay differential equations. *Numerical Algorithms*. 74(1), 223–245.

Ray, S. S. (2012). On Haar wavelet operational matrix of general order and its application for the numerical solution of fractional Bagley Torvik equation. *Applied Mathematics and Computation*. 218(9), 5239–5248.

Razzaghi, M. and Yousefi, S. (2001). The Legendre wavelets operational matrix of integration. *International Journal of Systems Science*. 32(4), 495–502.

Ross, B. (1975). A brief history and exposition of the fundamental theory of fractional calculus. In *Fractional calculus and its applications*. (pp. 1–36). Springer.

Saadatmandi, A. and Dehghan, M. (2010). A new operational matrix for solving fractional-order differential equations. *Computers and Mathematics with applications*. 59(3), 1326–1336.

Saeedi, H., Moghadam, M. M., Mollahasani, N. and Chuev, G. (2011). A CAS wavelet method for solving nonlinear Fredholm integro-differential equations of fractional order. *Communications in Nonlinear Science and Numerical Simulation*. 16(3), 1154–1163.

Sahu, P. K. and Saha Ray, S. (2015). Legendre wavelets operational method for the numerical solutions of nonlinear Volterra integro-differential equations system. *Applied Mathematics and Computation*. 256, 715–723.

Sakar, M. G. (2017). Iterative reproducing kernel Hilbert spaces method for Riccati differential equations. *Journal of Computational and Applied Mathematics*. 309, 163–174.

Sekar, R. C. G. and Murugesan, K. (2016). System of linear second order Volterra integro-differential equations using Single Term Walsh Series technique. *Applied Mathematics and Computation*. 273, 484–492.

Shah, F. A. and Abbas, R. (2015). Haar Wavelet Operational Matrix Method for the Numerical Solution of Fractional Order Differential Equations. *Nonlinear Engineering*. 4(4), 203–213.

Singh, C. S., Singh, H., Singh, V. K. and Singh, O. P. (2016). Fractional order operational matrix methods for fractional singular integro-differential equation. *Applied Mathematical Modelling*. 40(23), 10705–10718.

Sohrabi, S. (2011). Comparison Chebyshev wavelets method with BPFs method for solving Abel's integral equation. *Ain Shams Engineering Journal*. 2(3), 249–254.

Srivastava, H. M. and Junesang, C. (2012). *Zeta and q-Zeta functions and associated series and integrals*. Elsevier.

Tarasov, V. E. (2016). On chain rule for fractional derivatives. *Communications in Nonlinear Science and Numerical Simulation*. 30(1), 1–4.

Tariq, H. and Akram, G. (2016). Quintic spline technique for time fractional fourth-order partial differential equation. *Numerical Methods for Partial Differential Equations*.

Thorwe, J. and Bhalekar, S. (2012). Solving partial integro-differential equations using Laplace transform method. *American Journal of Computational and Applied Mathematics*. 2(3), 101–104.

Tohidi, E., Ezadkhah, M. and Shateyi, S. (2014). Numerical solution of nonlinear fractional Volterra integro-differential equations via Bernoulli polynomials. In *Abstract and Applied Analysis*, vol. 2014. Hindawi Publishing Corporation.

ur Rehman, M. and Khan, R. A. (2011). The Legendre wavelet method for solving fractional differential equations. *Communications in Nonlinear Science and Numerical Simulation*. 16(11), 4163–4173.

Wang, L., Ma, Y. and Meng, Z. (2014). Haar wavelet method for solving fractional partial differential equations numerically. *Applied Mathematics and Computation*. 227, 66–76.

Wang, Y. and Zhu, L. (2017). Solving nonlinear Volterra integro-differential equations of fractional order by using Euler wavelet method. *Advances in Difference Equations*. 2017(1), 27.

Wazwaz, A. M. (2013). The Volterra integro-differential forms of the singular Flierl-Petviashvili and the Lane-Emden equations with boundary conditions. *Romanian Journal of Physics*. 58, 685–693.

Yang, X., Xu, D. and Zhang, H. (2013). Crank–Nicolson/quasi-wavelets method for solving fourth order partial integro-differential equation with a weakly singular kernel. *Journal of Computational Physics*. 234, 317–329.

Yousefi, S. and Razzaghi, M. (2005). Legendre wavelets method for the nonlinear Volterra-Fredholm integral equations. *Mathematics and Computers* *in Simulation*. 70(1), 1–8.

Yuanlu, L. (2010). Solving a nonlinear fractional differential equation using Chebyshev wavelets. *Communications in Nonlinear Science and Numerical Simulation*. 15(9), 2284–2292.

Yusufolu (Agadjanov), E. (2009). Improved homotopy perturbation method for solving Fredholm type integro-differential equations. *Chaos, Solitons and Fractals: the interdisciplinary journal of Nonlinear Science, and Nonequilibrium and Complex Phenomena*. 41(1), 28–37.

Yüzbaşı, Ş. (2015). Numerical solutions of system of linear Fredholm–Volterra integro-differential equations by the Bessel collocation method and error estimation. *Applied Mathematics and Computation*. 250, 320–338.

Yüzbaşı, Ş. (2016). A collocation method based on Bernstein polynomials to solve nonlinear Fredholm–Volterra integro-differential equations. *Applied Mathematics and Computation*. 273, 142–154.

Zhu, L. and Fan, Q. (2012). Solving fractional nonlinear Fredholm integro-differential equations by the second kind Chebyshev wavelet. *Communications in Nonlinear Science and Numerical Simulation*. 17(6), 2333–2341.